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# USES OF THE SPECIAL TRIPLE PRODUCT $\mathbf{ab}^2$ OF EXTENSIVE QUANTITIES.

By J. V. COLLINS, Stevens Point, Wisconsin.

1. The study of the special triple product of extensive quantities where the second factor is repeated gives interesting results. Thus by means of this product the nature of the quaternion itself can be developed directly from the fundamental laws of multiplication. The author of the *Ausdehnungslehre* developed his subject by using the fundamental laws of operation as a starting point, but writers on quaternions pursue a widely different course.

2. Following out the idea of developing quaternions from the fundamental laws assumed to hold *for products of the reference units only*,[*] the writer used the special triple product $\mathbf{aB}^2$ where $\mathbf{a}$ and $\mathbf{B}$ are extensive quantities, $\mathbf{B}$ being a unit quantity, rather than the dual product $\mathbf{ab}$, or the more general triple product $\mathbf{abc}$. By Hamilton's law[†] $\mathbf{aB}^2=-\mathbf{a}$. Then $(\mathbf{aB})\mathbf{B}$ by the associative law should likewise equal $-\mathbf{a}$. Expressing both $\mathbf{a}$ and $\mathbf{B}$ as sums of components along three rectangular reference axes, we may write

$$\mathbf{a}=a_i\mathbf{i}+a_j\mathbf{j}+a_k\mathbf{k},$$

$$\mathbf{B}=B_i\mathbf{i}+B_j\mathbf{j}+B_k\mathbf{k}.$$

Then if $M_i$, $M_j$, $M_k$ are the respective minors of $m_i$, $m_j$, $m_k$ in the determinant whose respective rows are $m_i$, $m_j$, $m_k$, $a_i$, $a_j$, $a_k$, $B_i$, $B_j$, $B_k$,

$$(\mathbf{aB})\mathbf{B}=-\mathbf{a}=-(a_iB_i+a_jB_j+a_kB_k)\mathbf{B}+ \begin{vmatrix} M_j & M_k \\ B_j & B_k \end{vmatrix}\mathbf{i}+\begin{vmatrix} M_k & M_i \\ B_k & B_i \end{vmatrix}\mathbf{j}+\begin{vmatrix} M_i & M_j \\ B_i & B_j \end{vmatrix}\mathbf{k}.$$

For, one has no trouble in interpreting by means of the assumed laws for the products of the units such terms as $(\mathbf{ij})\mathbf{i}$, $(\mathbf{ij})\mathbf{j}$, etc. Then only the three terms which contain the product of all three reference units remain. But after rearranging the terms which contain these products by the commutative and associative laws, and combining, the coefficient of the product vanishes identically. The simplified product just given shows $-\mathbf{a}$ "analyzed" into two components, one along $\mathbf{B}$ and the other perpendicular to it.

If then $(\mathbf{aB})\mathbf{B}=-\mathbf{a}$ we may assume

$$\frac{-\mathbf{a}}{\mathbf{B}}=\mathbf{aB},$$

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[*] See Grassmann's Ausdehnungslehre of 1862, Engel ed., Arts. 37, 42.

[†] It is easy to prove that if the squares of the reference units equal negative unity, the square of any unit vector has the same value.

which evidently makes of $\mathbf{aB}$ a kind of multiplier or operator that can change $\mathbf{B}$ into $-\mathbf{a}$.

Let us now take the case of three unit quantities, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, of the same kind.

$\mathbf{AB}$ is an operator which changes $\mathbf{B}$ into $-\mathbf{A}$,
$\mathbf{BC}$ is an operator which changes $\mathbf{C}$ into $-\mathbf{B}$,
$\mathbf{CA}$ is an operator which changes $\mathbf{A}$ into $-\mathbf{C}$,
$\mathbf{BA}$ is an operator which changes $\mathbf{A}$ into $-\mathbf{B}$, etc.

Attempting to satisfy these conditions one is led directly to put $\mathbf{AB}=\mathbf{C}$, which by our assumed laws immediately gives

$$\mathbf{BC}=\mathbf{A},\quad \mathbf{CA}=\mathbf{B},\quad \mathbf{CB}=-\mathbf{A},\quad \mathbf{AC}=-\mathbf{B},\quad \mathbf{BA}=-\mathbf{C}.$$

These equations with those already assumed, viz., $\mathbf{A}^2=\mathbf{B}^2=\mathbf{C}^2=-1$, are the fundamental laws of quaternions. From them, as Tait states,* the whole subject can be developed. We have already seen that $\mathbf{aB}$ is an operator which changes $\mathbf{B}$ into $-\mathbf{a}$: an easy extension interprets the meaning of $(\mathbf{aB})\mathbf{c}$.

Thus starting from the fundamental laws of multiplication only, one is led naturally and directly to the fully developed subject of quaternions or to Macfarlane's *Algebra of Physics.* In this way unity in the form of presentation is secured for all branches of vector analysis. Moreover there is nothing in the preceding to limit the meaning to vectors. Thus $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{a}$, $\mathbf{b}$ may stand for points, lines, and planes and the results be interpreted. This method of developing the subject seems much preferable to the one followed by writers on quaternions who begin arbitrarily with either the "$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$" system of vectors, or with the conception of a quaternion as an operator which changes one vestor into another, these assumptions or definitions being followed later by an investigation into the fundamental laws of multiplication. Moreover this plan has the additional merit of showing just why Hamilton made the square of a unit vector negative rather than positive unity. For had the scalar law chosen been $\mathbf{A}^2=\mathbf{B}^2=\mathbf{C}^2=1$, $(\mathbf{aB})\mathbf{B}$ would have come out $-\mathbf{a}$ instead of $\mathbf{a}$ as called for by the associative law. Thus the associative law has to be given up in the algebra of physics (unless the factor $i$ is introduced) and one must write $(\mathbf{Ba})\mathbf{B}=\mathbf{a}$ instead of $(\mathbf{aB})\mathbf{B}=\mathbf{a}$. It should be added however that this does not deprive the algebra of physics of usefulness, for in some ways it is more practical than the quaternions it so closelv resembles.

3. We have just seen to what use the product $\mathbf{aB}^2$ can be put when its parts are used together. Let us next study to what use they can be put when employed independently.

Let us denote extensive quantities by clarendon or black faced type, unit quantities by capitals, tensors by italics, and a perpendicular to a vector in positive direction of angle (in plane of factors) by writing a line over it. Then (§2) let

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*Tait, Quaternions, Art. 71, 2nd ed.

$\mathbf{a}^{o}\,\mathbf{b}$=cosine part of product$-(\mathbf{ab})\mathbf{B}=ab.\cos\angle\,\mathbf{ab}\cdot\mathbf{B}$,

$\mathbf{a}\,{}^{i}\,\mathbf{b}$=sine part of product$-(\mathbf{ab})\mathbf{B}=ab.\sin\angle\,\mathbf{ab}.\bar{B}$,

$\mathbf{a}\,{}^{t}\,\mathbf{b}$=tangent function of $\mathbf{a}$ and $\mathbf{b}=ab.\tan\angle\,\mathbf{ab}.\bar{B}$,

and $\mathbf{a}\,{}^{c}\,\mathbf{b}$=cotangent function, $\mathbf{a}\,{}^{z}\,\mathbf{b}$=secant function, and $\mathbf{a}^{k}\,\mathbf{b}$=cosecant function of vectors $\mathbf{a}$ and $\mathbf{b}$. Evidently $\mathbf{a}\,{}^{c}\,\mathbf{b}$, $\mathbf{a}\,{}^{z}\,\mathbf{b}$ differ only by a numerical factor from $\mathbf{a}\,{}^{o}\,\mathbf{b}$, both being measured in same direction as $\mathbf{a}\,{}^{o}\,\mathbf{b}$, while $\mathbf{a}\,{}^{t}\,\mathbf{b}$ and $\mathbf{a}^{k}\,\mathbf{b}$ differ in the same way from $\mathbf{a}\,{}^{i}\,\mathbf{b}$ and are measured in the same direction as $\mathbf{a}\,{}^{i}\,\mathbf{b}$.

4. By means of the expressions of the last articles one can write vector equations in a form corresponding to the cartesian ones, as $\mathbf{r}=\mathbf{x}+\mathbf{y}$, or $\mathbf{r}=\mathbf{x}+\mathbf{y}+\mathbf{z}$. Thus

$$\mathbf{r}=\mathbf{Q}\,{}^{o}\mathbf{a}+\mathbf{Q}\,{}^{o}\mathbf{b}$$

is the equation of an ellipse where $\mathbf{Q}$ is an auxiliary vector making with $\mathbf{a}$ the so-called eccentric angle, and $\mathbf{a}$ and $\mathbf{b}$ are the semi-axes. Evidently $\mathbf{Q}$ can be understood, whereupon the equation reduces to

$$\mathbf{r}={}^{o}\mathbf{a}+{}^{o}\mathbf{b}.$$

Similarly $\mathbf{r}={}^{z}\mathbf{a}-{}^{c}\mathbf{b}$ is the equation of the hyperbola referred to its axes, $\mathbf{r}={}^{c\,2}\mathbf{a}-{}^{\underset{\cdot}{c}}\mathbf{a}$ that of the parabola, the dot under $c$ denoting that the vector length is measured in the complementary direction to that of $c$ unmarked, here along $\mathbf{b}$: The same notation is available for higher plane curves. Thus

$\mathbf{r}={}^{z}\mathbf{a}\,{}^{z}\mathbf{b}$ corresponds to $x^2(y^2-b^2)=a^2y^2$,

$\mathbf{r}={}^{t}\mathbf{p}+{}^{o}\mathbf{m}+\mathbf{p}+{}^{i}\mathbf{m}$ corresponds to $m^2y^2=(p-y)^2(x^2+y^2)$.

In space we have* $\mathbf{r}={}^{o}\mathbf{a}+{}^{o}\mathbf{b}+{}^{o}\mathbf{c}$, as the vector equation of the ellipsoid, where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are vector axes, and the auxiliary vector $\mathbf{Q}$, understood, is not written. Similarly

$\mathbf{r}={}^{z}\mathbf{A}(\mathbf{a}+{}^{o}\mathbf{b}+{}^{o}\mathbf{c})$ is the equation of the hyperboloid of two sheets,

$\mathbf{r}={}^{k}\mathbf{C}[{}^{k}\mathbf{C}({}^{o}\mathbf{a}+{}^{o}\mathbf{b})-{}^{o}\mathbf{c}]$ is the equation of the hyperboloid of one sheet,

$\mathbf{r}={}^{c\,2}\mathbf{c}+{}^{c}\mathbf{c}.\,{}^{z}\mathbf{a}({}^{z}\mathbf{C}.\mathbf{a}+{}^{o}\mathbf{b})$ is the equation of the hyperbolic paraboloid,

$\mathbf{r}=\mathbf{q}\,{}^{o}\mathbf{C}+\mathbf{q}\,{}^{c}\mathbf{C}(\mathbf{Q}\,{}^{o}\mathbf{a}+\mathbf{Q}\,{}^{o}\mathbf{b})$ is the equation of the elliptical cone,

where $\mathbf{q}$ is an auxiliary vector whose tensor is equal to that of $\mathbf{r}$, and $\mathbf{a}$ and $\mathbf{b}$ are the axes of the ellipse at a unit's distance from vertex. $\mathbf{r}=\mathbf{q}\,{}^{o}\mathbf{C}+\mathbf{q}\,{}^{k}\mathbf{C}(\mathbf{Q}\,{}^{o}\mathbf{a}+\mathbf{Q}\,{}^{o}\mathbf{b})$ is the equation of the elliptical cylinder. To get the normal to a central quadric, say the ellipsoid, one writes down the scalar product of the dyadic

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*The derivation of this and the following equations is omitted for want of space.

$$\frac{AA'}{a^2}+\frac{BB'}{b^2}+\frac{CC'}{c^2}$$

and the radius vector ${}^o\mathbf{a}+{}^o\mathbf{b}+{}^o\mathbf{c}$ which gives $\frac{{}^oA}{a}+\frac{{}^oB}{b}+\frac{{}^oC}{c}$. Having the normal it is easy to construct the equation of the tangent plane.

It may be remarked that if the preceding equations were written in one of the usual notations, as, for instance, that used by Gibbs, they would become very complicated, though the concepts represented, as we have seen are simple enough.

5. Neither the usual vector equations, those just given, nor the cartesian equations, possess exclusive merits over the others. Each in turn can be superior to the others for certain purposes. It is easier to construct the curves and surfaces from the equations just given than from the cartesian equations since one can use the trigonometric tables. The equations for the *plane* curves do not differ except in compactness and convenience of notation from the vector equations commonly given. But those for space of three dimensions are new forms. Writers on vector analysis use the $\phi$ function for quadrics, which introduces very compact expressions, but ones essentially scalar. The use of the above equations makes the treatment of solid space analogous to that of plane and connects vector analysis much more closely to cartesian analysis.

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## NOTE ON THE TOTIENT OF A NUMBER.

By G. A. MILLER.

Like the preceding paper, "On the Totitives of Different Orders," MONTHLY, Vol. XI, p. 129, this note aims to point out the advantages of employing the theory of groups in the proofs of some of the elementary theorems of number theory.

The number of natural numbers which do not exceed the positive integer $m$ and are prime to $m$ is denoted by $\phi(m)$ according to Gauss.* This function of $m$ is known by various names. English, French, and German writers respectively employ the following names: totient of $m$, indicator of $m$, Euler's $\phi$-function of $m$. If 1, $d_1$, $d_2$, ........., $d_\lambda$, $m$ are all the divisors of $m$ it is well known that

$$\phi(1)+\phi(d_1)+\phi(d_2)+\dots\dots\dots+\phi(d_\lambda)+\phi(m)=m.$$

The main object of this note is to show the connection between this formula and the theory of cyclic groups in a very elementary manner. Several closely related questions in number theory are also treated from the standpoint of group theory.

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*Gauss, *Disquisitiones Arithmeticae*, Art. 38.